Introduction to Algorithms

Chapter 9 : Medians and Order Statistics

XiangYang Li and Haisheng Tan[1]

School of Computer Science and Technology
University of Science and Technology of China (USTC)

Fall Semester 2025

Outline of Topics

Selection Problem

- This chapter addresses the problem of selecting the i-th order statistic from a set of n distinct numbers. We formally specify the selection problem as follows:
  Input: A set A of n (distinct) numbers and an integer i, with $1 \leq i \leq n$.
  Output: The element $x \in A$ that is larger than exactly $i - 1$ other elements of A.

Minimum and Maximum

- To determine the minimum of a set of n elements, a lower bound of comparisons is $n-1$.
- The following procedure selects the minimum from the array A, where A.length = n.

```
MINIMUM(A)
min = A[1]
for i = 2 to A.length do
    if min > A[i] then
        min = A[i]
return min
```

Simultaneous Minimum and Maximum

- In some applications, we must find both the minimum and the maximum of a set of n elements.
- A simple solution: find the minimum and maximum independently, using $n-1$ comparisons for each, for a total of $2n-2$ comparisons.
- In fact, we can find both the minimum and the maximum using at most $3\lfloor n/2 \rfloor$ comparisons.

Simultaneous Minimum and Maximum

MAX-MIN(A)

```
if A[1] > A[2] then min = A[2], max = A[1]
                else min = A[1], max = A[2]
for i = 2 to ⌊n/2⌋ do
    if A[2i − 1] > A[2i]
        then if A[2i] < min then min = A[2i]
             if A[2i − 1] > max then max = A[2i − 1]
        else if A[2i − 1] < min then min = A[2i − 1]
             if A[2i] > max then max = A[2i]
if n ≠ 2⌊n/2⌋ then if A[n] < min then min = A[n]
                   if A[n] > max then max = A[n]
return (min, max)
```

Simultaneous Minimum and Maximum

- Total number of comparisons:

  If n is odd, then we perform $3\lfloor n/2 \rfloor$ comparisons. If n is even, we perform 1 initial comparison followed by $3(n-2)/2$ comparisons, for a total of $3n/2-2$. Thus, in either case, the total number of comparisons is at most $3\lfloor n/2 \rfloor$.

Selection in Expected Linear Time

- A divide-and-conquer algorithm for the selection problem: RANDOMIZED-SELECT.
- The idea is to partition the input array recursively (as in quick-sort).
- The difference is that quick-sort recursively processes both sides of the partition, but RANDOMIZED-SELECT only works on one side of the partition.
- Quick-sort has an expected running time of $\Theta(n \log n)$, but the expected time of RANDOMIZED-SELECT is $\Theta(n)$.

RANDOMIZED-SELECT

```
RANDOMIZED-SELECT(A, p, r, i)
if p == r then
    return A[p]
q = RANDOMIZED-PARTITION(A, p, r)
k = q - p + 1
if i == k then
    return A[q] // the pivot value is the answer
if i < k then
    return RANDOMIZED-SELECT(A, p, q - 1, i)
else
    return RANDOMIZED-SELECT(A, q + 1, r, i - k)
```

RANDOMIZED-SELECT - Analysis

- The worst-case running time for RANDOMIZED-SELECT is $\Theta(n^2)$, even to find the minimum, because we could be extremely unlucky and always partition around the largest remaining element, and partitioning takes $\Theta(n)$ time.
- The expected running time for RANDOMIZED-SELECT is $\Theta(n)$.

RANDOMIZED-SELECT - Analysis

- The time required by RANDOMIZED-SELECT on an input array $A[p \dots r]$ of n elements is denoted by $T(n)$.
- We define indicator random variables X_k where $X_k = I\{$ the subarray $A[p \dots q]$ has exactly k elements $\}$. So we have $E[X_k] = \frac{1}{n}$, X_k has the value 1 for exactly one value of k, and it is 0 for all other k. When $X_k = 1$, two subarrays on which we might recurse have sizes $k-1$ and $n-k$

RANDOMIZED-SELECT - Analysis

$$\begin{aligned} T(n) &\leq \sum_{k=1}^{n} X_k (T(\max(k-1, n-k)) + O(n)) \\ &= \sum_{k=1}^{n} X_k T(\max(k-1, n-k)) + O(n) \end{aligned}$$

$$\begin{aligned} E[T(n)] &\leq E[\sum_{k=1}^{n} X_k T(\max(k-1, n-k)) + O(n)] \\ &= \sum_{k=1}^{n} E[X_k T(\max(k-1, n-k))] + O(n) \\ &= \sum_{k=1}^{n} E[X_k] E[T(\max(k-1, n-k))] + O(n) \\ &= \sum_{k=1}^{n} \frac{1}{n} E[T(\max(k-1, n-k))] + O(n) \end{aligned}$$

RANDOMIZED-SELECT - Analysis

$$\max(k-1, n-k) = \begin{cases} k-1 \text{ if } k > \lceil n/2 \rceil, \\ n-k \text{ if } k \leq \lceil n/2 \rceil \end{cases}$$

$$E[T(n)] \leq \frac{2}{n} \sum_{k=\lfloor n/2 \rfloor}^{n-1} E(T(k)) + O(n)$$

- Assume that $T(n) \leq cn$ for some constant c that satisfies the initial conditions of the recurrence. Pick a constant a such that the function described by the $O(n)$ term above (which describes the non-recursive component of the running time of the algorithm) is bounded from above by an for all $n > 0$.

RANDOMIZED-SELECT - Analysis

$$
\begin{aligned}
E[T(n)] &\le \frac{2}{n}\sum_{k=\lfloor n/2 \rfloor}^{n-1} ck + an \\
&= \frac{2c}{n}\left(\sum_{k=1}^{n-1} k - \sum_{k=1}^{\lfloor n/2 \rfloor -1} k\right) + an \\
&= \frac{2c}{n}\left(\frac{(n-1)n}{2} - \frac{(\lfloor n/2 \rfloor - 1)\lfloor n/2 \rfloor}{2}\right) + an \\
&\le \frac{2c}{n}\left(\frac{(n-1)n}{2} - \frac{(n/2-2)(n/2-1)}{2}\right) + an \\
&= c\left(\frac{3n}{4} + \frac{1}{2} - \frac{2}{n}\right) + an \\
&\le \frac{3cn}{4} + \frac{c}{2} + an = cn - \left(\frac{cn}{4} - \frac{c}{2} - an\right)
\end{aligned}
$$

RANDOMIZED-SELECT - Analysis

- For sufficiently large n, we have

$$n(\frac{c}{4} - a) \geq \frac{c}{2}$$

- As long as we choose the constant c so that $c/4 - a > 0$, i.e., $c > 4a$, we can divide both sides by $c/4 - a$, giving

$$n \geq \frac{c/2}{c/4 - a} = \frac{2c}{c - 4a}$$

- If we assume that $T(n) = O(1)$ for $n < \frac{2c}{c-4a}$, we have $T(n) = O(n)$.
- So any order statistic, and in particular the median, can be determined on average in linear time.

Selection in Worst-case Linear Time

- We now examine a selection algorithm whose running time is O(n) in the worst case. Like RANDOMIZED-SELECT, the algorithm SELECT finds the desired element by recursively partitioning the input array.
- The SELECT algorithm determines the i th smallest of an input array of $n > 1$ distinct elements by executing the following steps. (If $n = 1$, then SELECT merely returns its only input value as the i th smallest.)

Selection in Worst-case Linear Time

- 1. Divide the n elements of the input array into $\lfloor n/5 \rfloor$ groups of 5 elements each and at most one group made up of the remaining $n \mod 5$ elements.
- 2. Find the median of each of the $\lfloor n/5 \rfloor$ groups.
- 3. Use SELECT recursively to find the median x of the $\lfloor n/5 \rfloor$ medians found in step 2.
- 4. Partition the input array around the median-of-medians x using the modified version of PARTITION. So that x is the kth smallest element and there are $n-k$ elements on the high side and $k-1$ elements on the low side.
- 5. If $i=k$, then return x. Otherwise, use SELECT recursively to find the i th smallest element on the low side if $i<k$, or the $(i-k)$ th smallest element on the high side if $i>k$.

Selection in Worst-case Linear Time - Analysis

- To analyze the running time of SELECT, we first determine a lower bound on the number of elements that are greater than the partitioning element x.
- At least half of the $\lceil n/5 \rceil$ groups contribute 3 elements that are greater than x, except for the one group that has fewer than 5 elements if 5 does not divide n exactly, and the one group containing x itself. So the number of elements greater than x is at least

$$3\left(\left\lceil \frac{1}{2}\left\lceil \frac{n}{5} \right\rceil \right\rceil - 2\right) \geq \frac{3n}{10} - 6$$

- So in the worst case, SELECT is called recursively on at most $\frac{7n}{10} + 6$ elements in step 5.

Selection in Worst-case Linear Time - Analysis

- Steps 1, 2, and 4 take $O(n)$ time.
- Step 3 takes time $T(\lceil n/5 \rceil)$, and step 5 takes time at most $T(7n/10+6)$, assuming that T is monotonically increasing
- Assume that any input of 140 or fewer elements requires $O(1)$ time.
- So we have the recurrence

$$T(n) \leq \begin{cases} \Theta(1) & \text{if } n \leq 140, \\ T(\lceil n/5 \rceil) + T(7n/10+6) + O(n) & \text{if } n > 140. \end{cases}$$

Selection in Worst-case Linear Time - Analysis

- Assuming that $T(n) \le cn$ for some suitably large constant c and all $n \le 140$.
- Pick a constant a such that the function described by the $O(n)$ term above is bounded above by an for all $n > 0$.
- So we have

$$\begin{aligned} T(n) &\le c\lceil n/5\rceil + c(7n/10+6) + an \\ &\le cn/5 + c + 7cn/10 + 6c + an \\ &= 9cn/10 + 7c + an \\ &= cn + (-cn/10 + 7c + an) \end{aligned}$$

Selection in Worst-case Linear Time - Analysis

- Thus T(n) is at most cn if($-cn/10 + 7c + an \leq 0$)

  $$c \geq 10a(n/(n-70)) \text{when } n > 70$$

  Because $n \geq 140$ $\quad n/(n-70) \leq 2$
  So choosing $c \geq 20a$ will satisfy inequality.
- The worst-case running time of SELECT is therefore linear.
- The algorithm is still correct if each group has r elements where r is odd and is not less than 5.

Selection in Worst-case Linear Time - Analysis

- Sorting requires $\Omega(n \log n)$ time in the comparison model, even on average, and the linear-time sorting algorithms in Chapter 8 make assumptions about the input.
- But the linear-time selection algorithms in this chapter do not require any assumptions about the input.
- The running time is linear because these algorithms do not sort.