Introduction to Algorithms

Chapter 23 : Minimum Spanning Trees

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Outline of Topics

Basic definitions and properties

In this chapter, we shall examine two algorithms for solving the minimum spanning-tree problem: Kruskal's algorithm and Prim's algorithm.

- Section 23.1 introduces a "generic" minimum-spanning-tree method that grows a spanning tree by adding one edge at a time.
- Section 23.2 gives two algorithms that implement the generic method.

Basic definitions and properties

Definition 1:

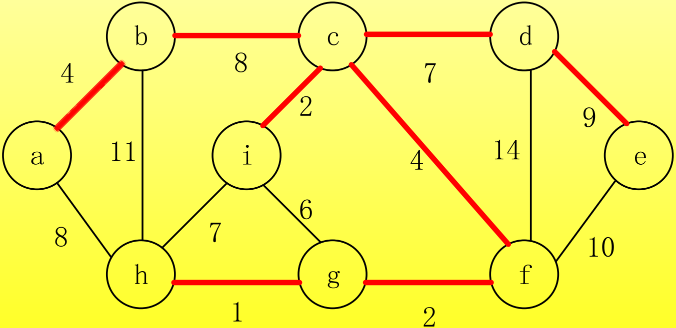
Given a connected, undirected graph $G = (V, E)$, for each edge $(u, v) \in E$, having a weight $w(u, v)$.

We wish to find an acyclic subset $T \subseteq E$ that connects all of the vertices and whose total weight is minimized.

$$w(T) = \sum_{(u,v)\in T} w(u, v)$$

Since T is acyclic and connects all of the vertices, it must form a tree, which we call a spanning tree since it “spans” the graph G. We call the problem of determining the tree T the minimum-spanning-tree problem(MST).

Example of a connected graph and MST



Greedy Method for MST

The two algorithms (Kruskal's Algorithm and Prim's Algorithm) we consider in this chapter run in time $O(|E|log|V|)$ using **a greedy approach** to the problem.

Greedy strategy:

Grows the minimum spanning tree one edge at a time and manages a set of edges A, maintaining the following loop invariant:

Prior to each iteration, A is a subset of some minimum spanning tree.

Generic MST Algorithm

At each step we determine an edge (u, v) such that $A \cup (u, v)$ is still a subset of a MST and (u, v) is called a **safe edge** for A.

GENERIC-MST(G, w)

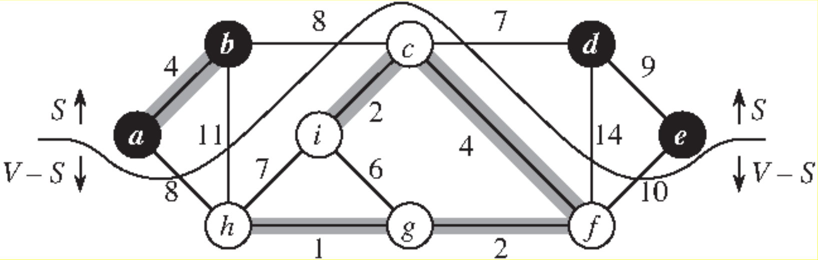
$A = \emptyset$
while A does not form a spanning tree **do**
  find an edge (u, v) that is safe for A
  $A = A \cup (u, v)$
return A

Cut and Light Edge

Definition 2:

- A **cut** $(S, V-S)$ of an undirected graph $G=(V,E)$ is a partition of V.
- An edge $(u,v) \in E$ crosses the cut iff one of its endpoints is in S and the other is in $V-S$
- An edge is a **light edge** crossing a cut if its weight is the minimum of any edge crossing the cut
- We call a cut **respects** a set A of edges if no edge in A crosses the cut.

Example of Cut and Light Edge



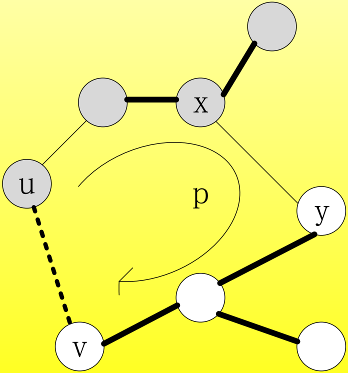
Recognize safe edges

Theorem 23.1:

Let $G = (V, E)$ be a connected, undirected graph with a real-valued weight function w defined on E. Let A be a subset of E that is included in some minimum spanning tree for G, let $(S, V - S)$ be any cut of G that respects A, and let (u, v) be a light edge crossing $(S, V - S)$.

Then, edge (u, v) is safe for A.

Recognize safe edges



Recognize safe edges

Proof:

Suppose that T is an MST containing A but not the light edge (u,v), then there is at least one edge on the path p that crosses the cut, say(x,y).

1.form a new spanning tree T':

Then (x,y) is not in A. Because p is the unique path from u to v in T, so removing (x,y) breaks T into two components. Adding (u,v) reconnects them to form a new spanning tree
$T' = (T - \{(x,y)\}) \cup \{(u,v)\}$.

Recognize safe edges

2. show that T' is MST:

Since (u,v) is a light edge crossing $(S,V-S)$ and (x,y) also crosses this cut,$w(u,v) \leq w(x,y)$. Therefore,

$$\begin{aligned} w\left(T'\right) &= w(T) - w(x,y) + w(u,v) \\ &\leq w(T) \end{aligned}$$

But T is a minimum spanning tree, so that $w(T) \leq w(T')$.thus T' must be a minimum spanning tree also.

3. show that (u,v) is a safe edge for A:

We have $A \subseteq T'$, since $A \subseteq T$ and $(x,y) \notin A$,thus $A \cup \{(u,v)\} \subseteq T'$. Since T' is a MST, (u,v) is safe for A.

Corollary to Theorem 23.1

Corollary to theorem 23.1:

Let $G = (V, E)$ be a connected, undirected graph with a real-valued weight function w defined on E. Let A be a subset of E that is included in some minimum spanning tree for G, and let $C = (V_c, E_c)$ be a connected component (tree) in the forest $G_A = (V, A)$.

If (u, v) is a light edge connecting C to some other component in G_A, then (u, v) is safe for A.

Proof:

Since the cut $(V_c, V - V_c)$ respects A, and (u, v) is a light edge for this cut, (u, v) is safe for A.

Kruskal and Prim Algorithms

- In **Kruskal's algorithm**, the set A forms a forest. The safe edge added to A is always a least-weight edge in the graph that connects two distinct components
- In **Prim's algorithm**, the set A forms a single tree. The safe edge added to A is always a least-weight edge connecting the tree to a vertex not in the tree.

Kruskal's algorithm

Kruskal's algorithm finds a safe edge to add to the growing forest by finding, of all the edges that connect any two trees in the forest, an edge (u,v) of **the least weight**.

Let C_1 and C_2 denote the two trees that are connected by (u,v). Since (u,v) is a light edge, connecting C_1 to some other tree, (u,v) is a safe edge for C_1.(Corollary 23.2)

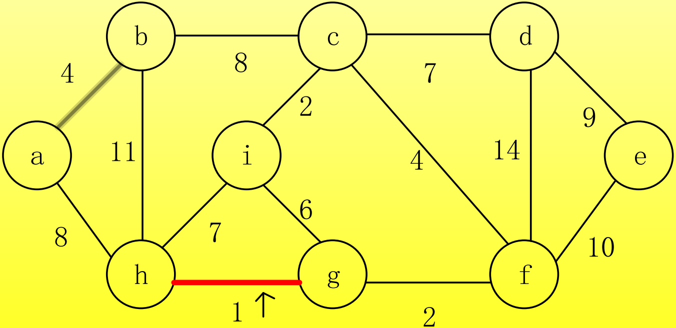
Simply speaking, at each step Kruskal's algorithm adds to the forest an edge of the least possible weight (greedy).

Kruskal's algorithm

MST-KRUSKAL(G, w)

$A = \emptyset$
for each vertex $v \in G.V$ **do**
    MAKE-SET(v)
sort the edges of $G.E$ into nondecreasing order by weight w
for each edge $(u, v) \in G.E$, taking in nondecreasing order by weight w, **do**
    if FIND-SET(u) $\neq$ FIND-SET(v) **then**
        $A = A \cup \{(u, v)\}$
        UNION(u, v)
return A

Kruskal's algorithm



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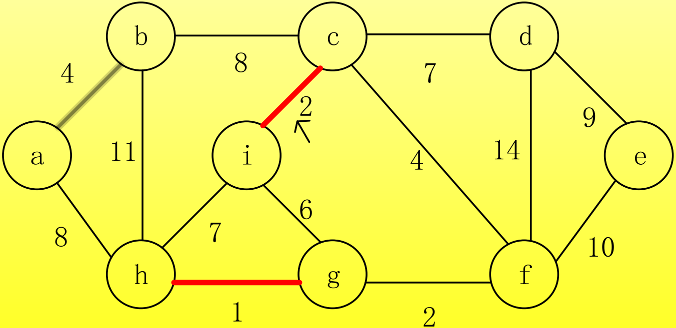

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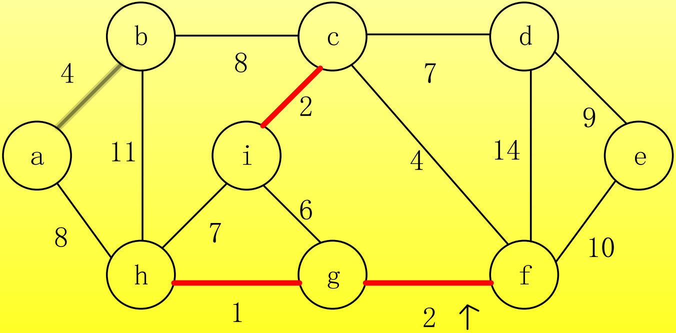

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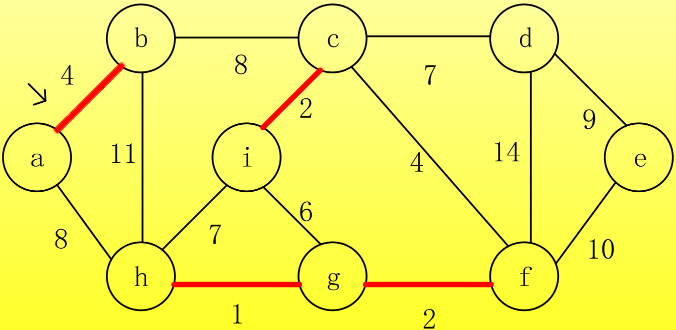

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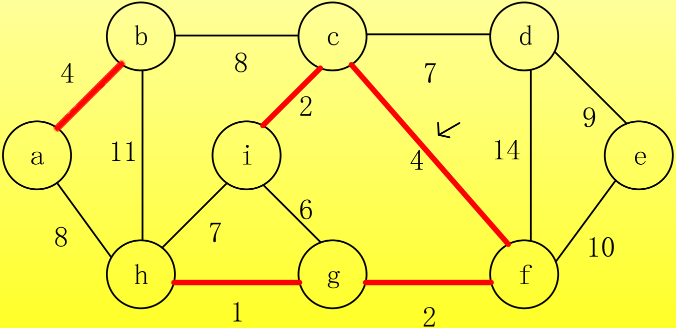

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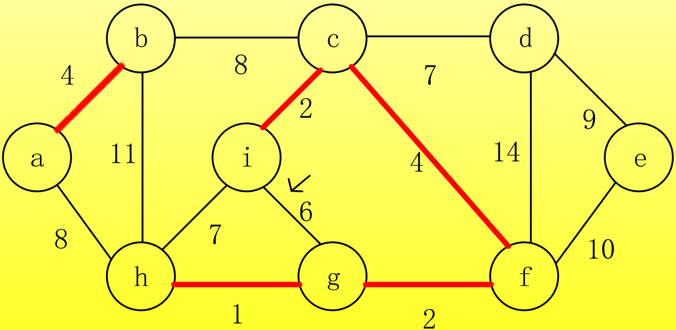

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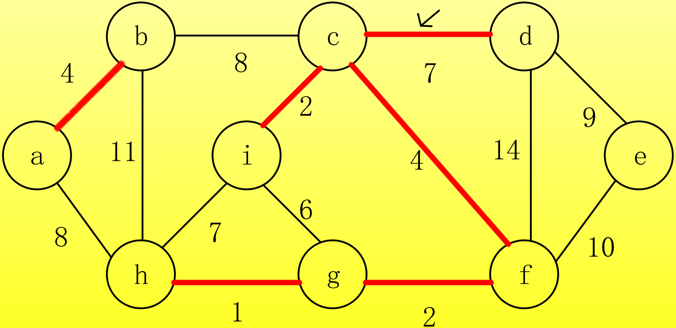

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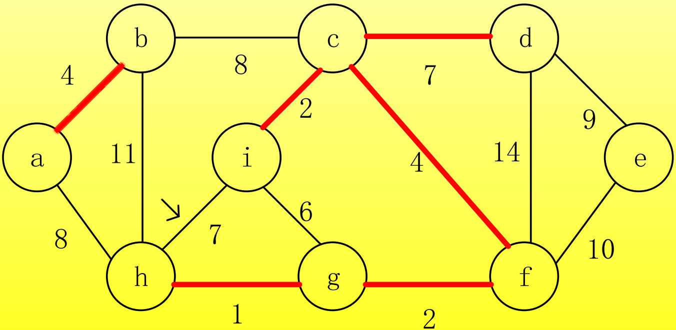

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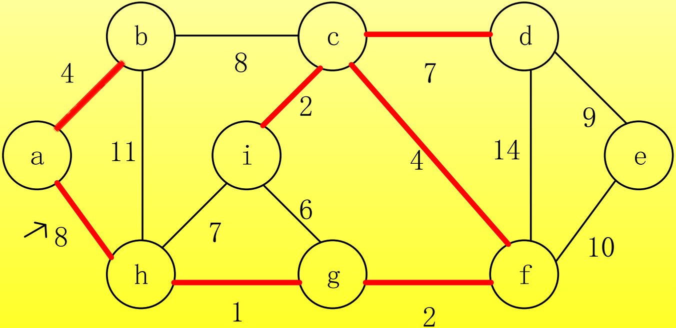

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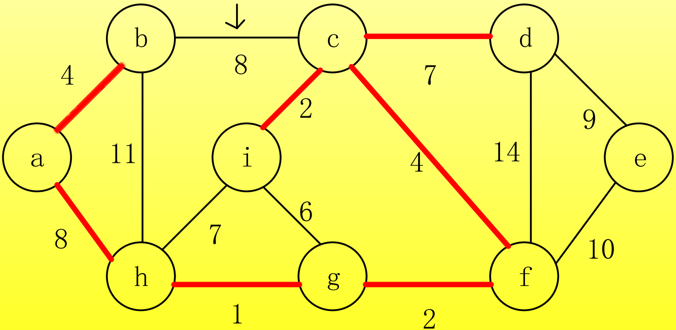

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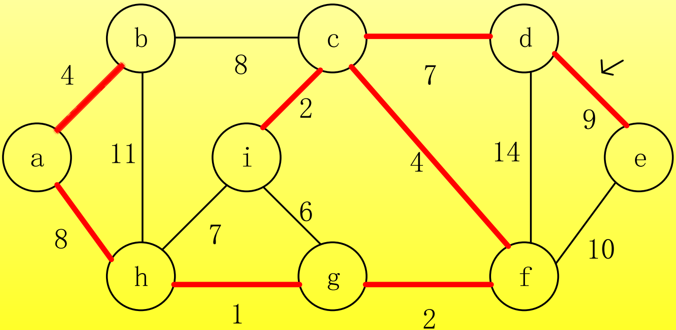

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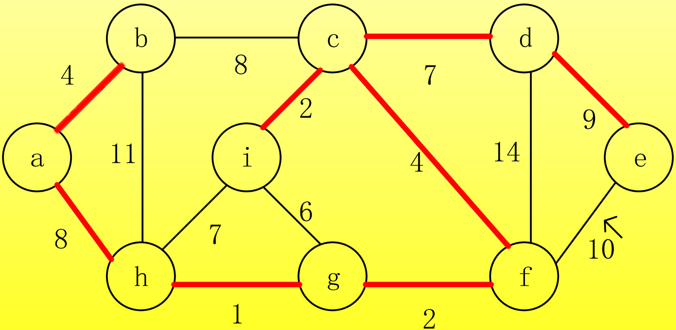

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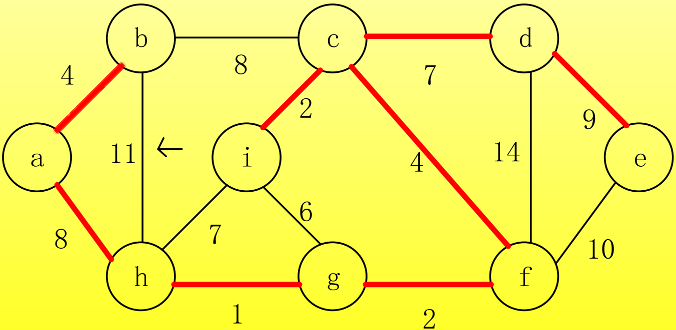

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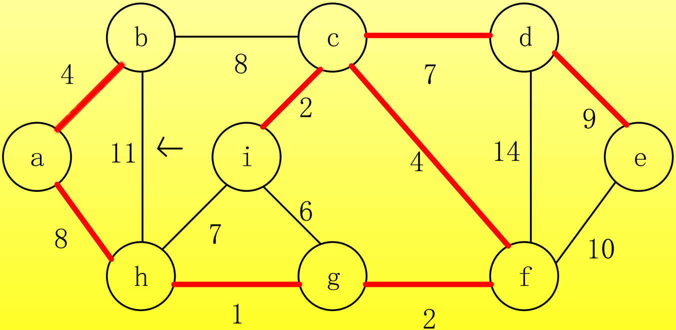

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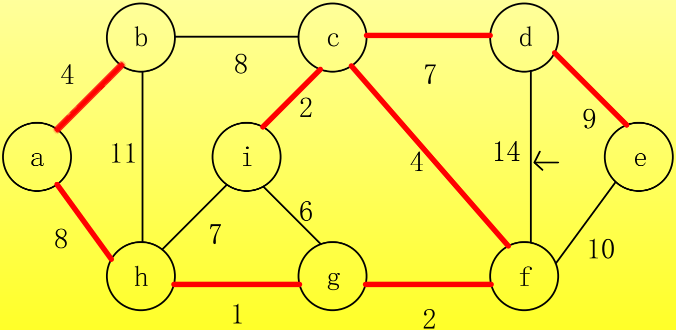


Kruskal's algorithm

MST-KRUSKAL(G, w)

$A = \emptyset$
for each vertex $v \in G.V$ **do**
    MAKE-SET(v) // $O(V)$ MAKE-SET
sort the edges of $G.E$ into nondecreasing order by weight w // $O(E \log E)$
for each edge $(u, v) \in G.E$, taking in nondecreasing order by weight w, **do**
    if FIND-SET(u) $\neq$ FIND-SET(v) **then**
        $A = A \cup \{(u, v)\}$
        UNION(u, v) //totally $O(E)$ FIND-SET and UNION
return A

Kruskal's Algorithm Complexity

Complexity:

Assume that we use the disjoint-set-forest implementation with the union-by-rank and path-compression heuristics

Initializing the set A in line 1 takes$O(1)$ time

Sort the edges in line 4 is $O(E \lg E)$ time

The for loop of lines 5-–8 performs $O(E)$ FIND-SET and UNION operations on the disjoint-set forest

Along with the $|V|$ MAKE-SET operations, these take a total of $O((V+E)\alpha(V))$ time

Since $\alpha(|V|) = O(\lg V) = O(\lg E)$, the running time is $O(E \lg E)$

Since $|E| \leq |V|^2$, $\lg |E| = O(\lg V)$, the running time is $O(E \lg V)$

Prim's algorithm

Prim's algorithm operates much like Dijkstra's algorithm for finding shortest paths in a graph.

Prim's algorithm has the property that the edges in the set A **always form a single tree.**

Each step adds to the tree A a light edge that connects A to an isolated vertex – one on which no edge of A is incident.

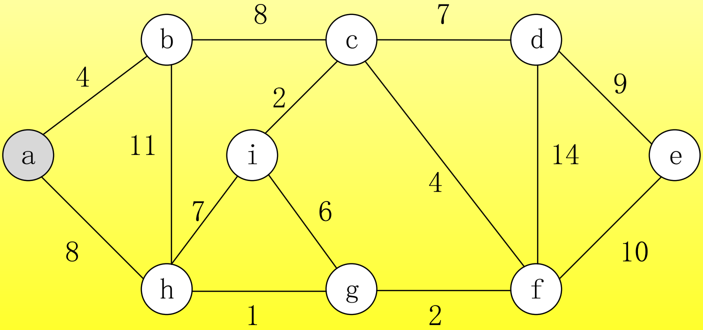
Simply speaking, at each step it adds to the tree an edge that contributes the minimum amount possible to the tree's weight (greedy).

Prim's algorithm

MST-PRIM(G, w, r)

for each vertex $u \in G.V$ **do**
  $u.key = \infty$; //$u.key$ stores the minimum weight of any edge connecting u to a vertex in the current tree
  $u.\pi = NIL$
$r.key = 0$
$Q = G.V$ // Q contains nodes not yet joining the tree
while $Q \neq \emptyset$ **do**
  u = EXTRACT-MIN(Q) //adding $(u.\pi, u)$ to the tree
  for each $v \in G.Adj[u]$ **do**
    if $v \in Q$ and $w(u, v) < v.key$ **then** //updating keys
      $v.\pi = u$
      $v.key = w(u, v)$

Prim's algorithm



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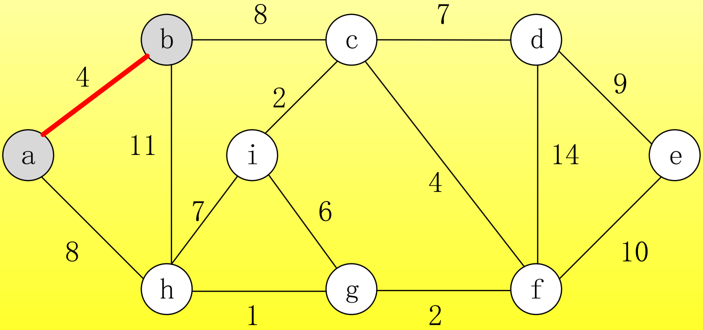

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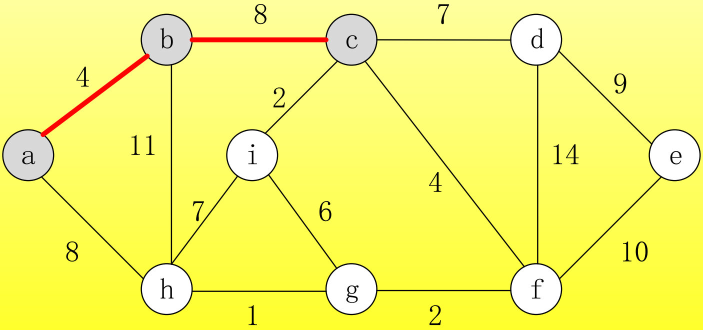

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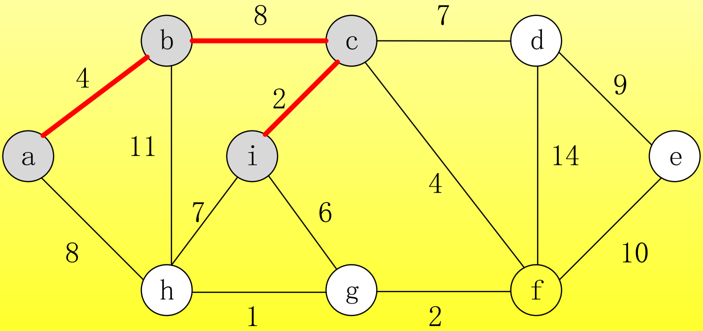

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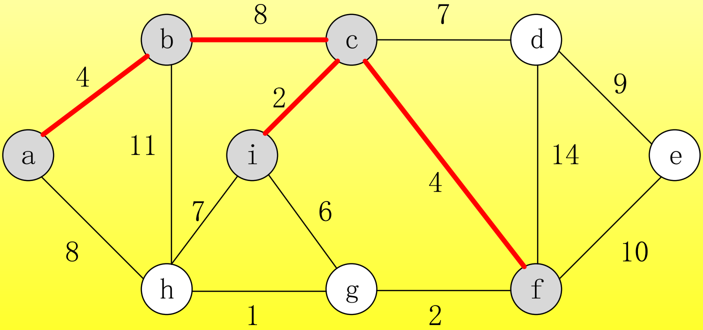

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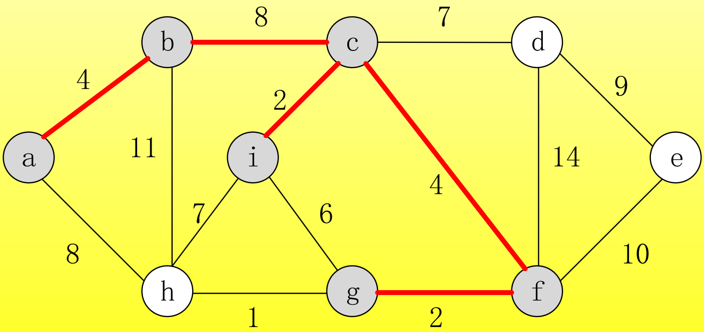

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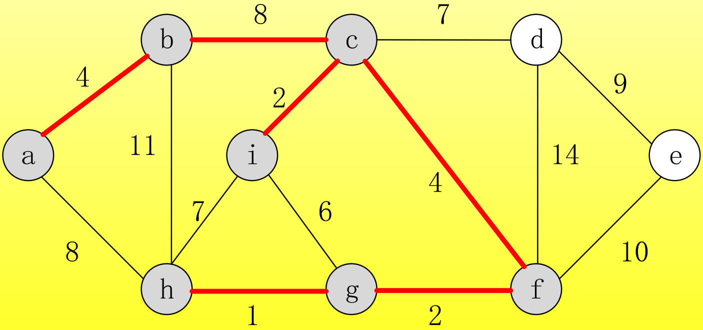

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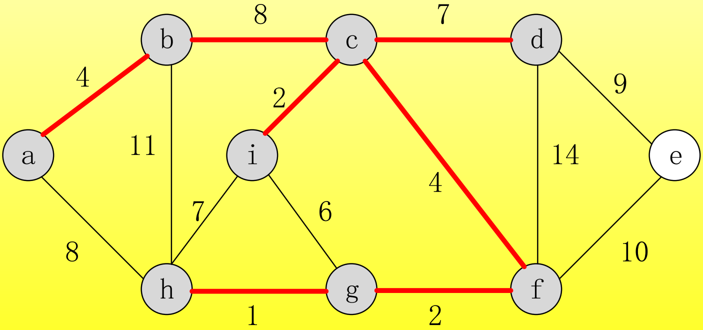

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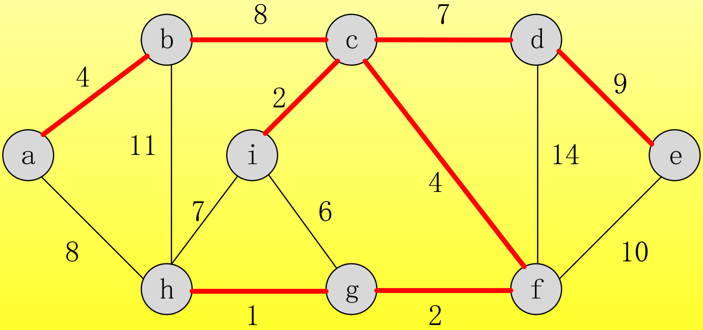

Prim's algorithm

MST-PRIM(G, w, r)

for each vertex $u \in G.V$ **do**
  $u.key = \infty$
  $u.\pi = NIL$
$r.key = 0$
$Q = G.V$ //BUILD-MIN-HEAP, $O(V)$
while $Q \neq \emptyset$ **do**   //V loops
  u = EXTRACT-MIN(Q)   //$O(\log V)$ for each loop
  for each $v \in G.Adj[u]$ **do** //   //2E loops totally
    if $v \in Q$ and $w(u, v) < v.key$ **then**
      $v.\pi = u$
      $v.key = w(u, v)$ //DECREASE-KEY

Prim's algorithm Complexity

Complexity:

Implement the min-priority queue Q as a binary min-heap:

Lines 1 - 5 : use the BUILD-MIN-HEAP to perform $O(V)$

The body of the while loop executes $|V|$ times, since each EXTRACT-MIN operation takes $O(\lg V)$ time, the total time if $O(VlgV)$ time. The for loop in lines 8 - 11 executes $O(E)$ times altogether, since the sum of the lengths of all adjacency lists is $2|E|$.

Line 11 involves an implicit DECREASE-KEY operation on the min-heap, which a binary min-heap supports in $O(\lg V)$ time.

Total time: $O(V \lg V + E \lg V) = O(E \lg V)$

What about implementing the min-priority queue Q as a FIB-Heap?